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Deconstructing plane anisotropic elasticity Part II: Stroh's formalism sans frills

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Abstract

Eigensolutions for all types of anisotropic elastic materials are obtained in terms of the eigenvalues and the anisotropic elastic stiffness. The generalized eigenvectors and eigensolutions in the degenerate and extra-degenerate cases are obtained by the derivative rule. A complete set of *unnormlized* eigenvectors, now given in terms of the elastic moduli, define the Barnett–Lothe tensors by the same expressions irrespective of material degeneracy. Explicit expressions of the Barnett–Lothe tensors are obtained in various forms depending on the multiplicity of eigenvalues. These expressions complement the alternative expressions of Part I in terms of the elastic compliances. A new family of extra-degenerate materials is found, suggesting the superabundance of such materials. A concise proof of the equivalence of the eigensystems of the compliance-based and elasticity-based formalisms is given. Eigenrelations applicable to all cases of material degeneracy are presented in both three-dimensional and six-dimensional matrix formalisms. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Lekhnitskii (1963) showed that the general solutions of plane anisotropic elasticity may be represented by analytic functions of the complex variables $x + \mu_i y$, where the μ_i's are the roots of a characteristic equation depending on the elastic compliances. The representation allows the differential equations governing the eigensolutions of displacements and stress potentials to be reduced to algebraic equations for the corresponding eigenvectors. An alternative analysis approach, developed by Stroh (1958) and others, yields the eigensolutions in terms of the anisotropic elastic moduli. While the results given in the

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earlier works were restricted to the case when the six roots of the characteristic equation are all distinct, later work by Ting and Hwu (1988) presents the eigensolutions for materials that are degenerate but not extra-degenerate.

In Part I of this paper, a systematic procedure was developed to obtain explicit expressions of the eigensolutions for all types of anisotropic materials, whether nondegenerate, degenerate or extra-degenerate. When the number of independent eigenvectors is smaller than the multiplicity of an eigenvalue, additional eigensolutions must be found in terms of *generalized* eigenvectors and the latter are determined by eigenrelations different from those governing the usual eigenvectors. The generalized eigenvectors and eigensolutions may be obtained by the derivative rule, as shown in Part I for the Lekhnitskii formalism and in the following for the Stroh formalism.

In Part II of this paper, the eigensolutions and the Barnett–Lothe tensors for all types of anisotropic materials are obtained in terms of the elastic moduli, i.e., the elements of $[\beta]^{-1}$. In this approach, the displacement eigenvectors (the **a**-vectors) are to be determined from a 3×3 eigenmatrix $\Gamma(\mu)$ whose elements are quadratic functions of the eigenvalue μ. In contrast, the approach in Part I was based on a 2×2 eigenmatrix $\mathbf{M}(\mu)$ governing the last two components of the **b**-vectors. This asymmetry in the dual formalism contributes to differences in the algebraic analysis and in the formal expressions of the results. However, the two formalisms yield identical eigenvalues and equivalent systems of eigenvectors and generalized eigenvectors. It is shown, as in Part I, that *unnormalized* eigenvectors and generalized eigenvectors satisfy (modified) orthogonality and closure relations, so that the Barnett–Lothe tensors may be defined in the same manner regardless of material degeneracy. Identities involving these tensors, some well known and others new, are shown to be the direct consequences of such definitions, and, therefore, are also valid regardless of material degeneracy.

2. Eigenrelations in the dual formalism

An eigensolution for the displacement, stress potential, strain and stress is given by the following expressions in terms of a complex eigenvalue μ, a pair of eigenvectors $\mathbf{a} = \{a_1, a_2, a_3\}$ and $\mathbf{b} = \{b_1, b_2, b_3\}$, and an arbitrary analytic function f:

$$\mathbf{u} = \mathbf{a}f(x + \mu y), \qquad \mathbf{q} = \boldsymbol{b}f(x + \mu y)$$

$$\{\epsilon\} = \mathbf{E}(\mu)\mathbf{a}f'(x + \mu y), \qquad \{\sigma\} = \sum \mathbf{P}(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} f'(x + \mu_i y), \tag{1}$$

where $\mathbf{u} = \{u, v, w\}^{\mathrm{T}}$, $\{\mathbf{q}\} = \{F_y, -F_x, \Psi\}^{\mathrm{T}}$, $\{\epsilon\} = \{\epsilon_x, \epsilon_y, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\}^{\mathrm{T}}$, $\{\sigma\} = \{\sigma_x, \sigma_y, \tau_{yz}, \tau_{xz}, \tau_{xy}\}^{\mathrm{T}} = \{F_{yy}, F_{xx}, -\Psi_x, \Psi_y, -F_{xy}\}^{\mathrm{T}}$ and

$$\mathbf{E}(\mu) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \\ 0 & 0 & 1 \\ \mu & 1 & 0 \end{bmatrix}, \quad \mathbf{P}(\mu) = \begin{bmatrix} -\mu^2 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & \mu \\ \mu & 0 \end{bmatrix} \tag{2a,b}$$

The first two components of the **b**-vector are related by $b_1 = -\mu b_2$ because $\tau_{xy} = -\partial_x F_y = -\partial_y F_x$. Substituting the third and fourth expressions of Eq. (1) into the anisotropic stress–strain relation $\{\epsilon\} = [\beta]\{\sigma\}$, one obtains the eigenrelation

$$\mathbf{E}(\mu)\mathbf{a} = [\beta]\mathbf{P}(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} \tag{3}$$

The matrices $\mathbf{E}$ and $\mathbf{P}$ of Eq. (2a,b) satisfy the identity $\mathbf{E}^{\mathrm{T}}\mathbf{P} = \mathbf{P}^{\mathrm{T}}\mathbf{E} = 0$. Hence Eq. (3) yields the equations governing the eigenvectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{E}^{\mathrm{T}}(\mu)[\beta]^{-1}\mathbf{E}(\mu)\mathbf{a} \equiv \Gamma(\mu)\mathbf{a} = \mathbf{0}, \tag{4}$$

$$\mathbf{P}^{\mathrm{T}}(\mu)[\beta]\mathbf{P}(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} \equiv \mathbf{M}(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} = \mathbf{0}, \tag{5}$$

as well as the characteristic equation determining the eigenvalue μ:

$$\Delta(\mu) \equiv |\Gamma(\mu)| = 0 \quad \text{or} \quad \delta(\mu) \equiv |\mathbf{M}(\mu)| = 0. \tag{6a,b}$$

Eq. (3) and the relation $b_1 = -\mu b_2$ imply the transformation rule between the **a-** and **b-**vectors:

$$\mathbf{a} \equiv \mathbf{Y}^{\mathrm{T}}(\mu)[\beta]\begin{bmatrix} 0 & -\mu^2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{bmatrix}\mathbf{b}, \qquad \mathbf{b} = -\mathrm{d}\mathbf{E}^{\mathrm{T}}(\mu)/\mathrm{d}\mu[\beta]^{-1}\mathbf{E}(\mu)\mathbf{a} \tag{7a,b}$$

and

$$\mathbf{b} = \mathbf{J}(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} \tag{7c}$$

where the matrices

$$\mathbf{J}(\mu) \equiv \begin{bmatrix} -\mu & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{Y}(\mu) \equiv \begin{bmatrix} 1 & -\mu & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{8a,b}$$

have the properties $\mathbf{YJ}=\mathrm{d}\mathbf{P}/\mathrm{d}\mu$, $\mathbf{J}=\mathbf{E}^{\mathrm{T}}\mathrm{d}\mathbf{P}/\mathrm{d}\mu$ and $\mathbf{Y}^{\mathrm{T}}\,\mathbf{E}$ is the 3×3 identity matrix. We write

$$[\beta]^{-1} = \begin{bmatrix} C_{11} & C_{12} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{24} & C_{25} & C_{26} \\ C_{14} & C_{24} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

and define

$$\mathbf{Q} = \begin{bmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{26} & C_{46} \\ C_{56} & C_{25} & C_{45} \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{bmatrix} \tag{9a,b,c}$$

Then Eq. (2a,b) gives the following expressions of the eigenmatrices in Eqs. (4) and (5):

$$\Gamma(\mu) \equiv \mathbf{E}^{\mathrm{T}}(\mu)[\beta]^{-1}\mathbf{E}(\mu) = \mathbf{Q} + \mu(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + \mu^2\mathbf{T}, \tag{10}$$

$$\mathbf{M}(\mu) \equiv \mathbf{P}^{\mathrm{T}}(\mu)[\beta]\mathbf{P}(\mu) = \begin{bmatrix} l_4(\mu) & -l_3(\mu) \\ -l_3(\mu) & l_2(\mu) \end{bmatrix} \tag{11}$$

where

$$l_4(\mu) = \beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22}$$

$$l_3(\mu) = \beta_{15}\mu^3 - (\beta_{14} + \beta_{56})\mu^2 + (\beta_{25} + \beta_{46})\mu - \beta_{24}$$

$$l_2(\mu) = \beta_{55}\mu^2 - 2\beta_{45}\mu + \beta_{44} \tag{12}$$

Since the eigenvalues cannot be real (Lekhnitskii, 1963), and in particular cannot be zero, the equation $\Gamma(0)\mathbf{a} = \mathbf{Qa} = \mathbf{0}$ has no nontrivial solution for $\mathbf{a}$. Hence $\mathbf{Q}$ is nonsingular. The matrix $\mathbf{T}$ may be obtained from $\mathbf{Q}$ by interchanging the x- and y-coordinate directions and rearranging the first pair of rows and columns in the resulting matrices. Therefore, $\mathbf{T}$ also cannot be singular.

Indeed $\mathbf{Q}$ and $\mathbf{T}$ are positive definite (Ting, 1996a). For a strain state with$\epsilon_y = \epsilon_z = \gamma_{yz} = 0$, $\epsilon = [\beta]\{\sigma\}$ yields $\{\sigma_x, \tau_{xy}, \tau_{xz}\}^{\mathrm{T}} = \mathbf{Q}\{\epsilon_x, \gamma_{xy}, \gamma_{xz}\}^{\mathrm{T}}$ and the strain energy density is $\frac{1}{2}\{\epsilon_x, \gamma_{xy}, \gamma_{xz}\}\mathbf{Q}\{\epsilon_x, \gamma_{xy}, \gamma_{xz}\}^{\mathrm{T}}$. Hence $\mathbf{Q}$ must be positive definite and a similar argument applies also to $\mathbf{T}$.

We next show that the matrix Γ never vanishes. Consider

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \mu \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & 1 & 0 \\ 0 & \mu & 0 & 0 & 1 \end{bmatrix} [\beta]^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \mu \\ 0 & 0 & 1 & \mu & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \mu & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_{11}(\mu) & C_{12} + \mu C_{26} & C_{14} + \mu C_{46} & \Gamma_{13}(\mu) & \Gamma_{12}(\mu) \\ C_{12} + \mu C_{26} & C_{22} & C_{24} & C_{25} + \mu C_{24} & C_{26} + \mu C_{22} \\ C_{14} + \mu C_{56} & C_{24} & C_{44} & C_{45} + \mu C_{44} & C_{46} + \mu C_{24} \\ \Gamma_{13}(\mu) & C_{25} + \mu C_{24} & C_{45} + \mu C_{44} & \Gamma_{33}(\mu) & \Gamma_{23}(\mu) \\ \Gamma_{12}(\mu) & C_{26} + \mu C_{22} & C_{46} + \mu C_{24} & \Gamma_{23}(\mu) & \Gamma_{22}(\mu) \end{bmatrix}$$

If all elements$\Gamma_{ij}(\mu)$ vanish for some μ, then the matrix on the right hand side is singular because its first, fourth and fifth column vectors cannot be all independent. This result contradicts the premise that $[\beta]^{-1}$ is non-singular. Therefore $\Gamma(\mu)$ has at least one nonvanishing column vector.

For any root μ_0 of the characteristic equation, the matrix $\Gamma(\mu_0)$ is singular and therefore it has at most two independent column vectors. If $\Gamma(\mu_0)$ has only one independent column vector for some eigenvalue μ_0 (this means that all elements of the adjoint matrix $\hat{\Gamma}(\mu_0)$ vanish), then Eq. (4) has two independent solutions $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$ and the corresponding $\mathbf{b}$-vectors $\boldsymbol{b}^{(i)} = -\mathbf{E}'(\mu_0)^{\mathrm{T}}[\beta]^{-1}\mathbf{E}(\mu_0)\mathbf{a}^{(i)}$ are also independent, because otherwise $\mathbf{a}^{(i)} = \mathbf{Y}(\mu_0)^{\mathrm{T}}[\beta]\mathbf{P}(\mu_0)\{b_2^{(i)}, b_3^{(i)}\}^{\mathrm{T}}(i = 1, 2)$ cannot be independent. Consequently, the equation $\mathbf{M}(\mu_0)\{b_2^{(i)}, b_3^{(i)}\}^{\mathrm{T}} = 0$ has two independent solutions. This implies that $\mathbf{M}(\mu_0) = \mathbf{0}$, so that the material is *abnormal*. We have shown that abnormal materials, originally defined in terms of $[\beta]$, have an equivalent characterization in terms of $[\beta]^{-1}$. The two equivalent conditions may be rephrased in the following way: a material is abnormal if and only if the adjoint matrix of $\mathbf{M}(\mu) = \mathbf{P}^{\mathrm{T}}(\mu)[\beta]\mathbf{P}(\mu)$ vanishes for some complex number μ; this is so if and only if the adjoint matrix of $\Gamma(\mu) = \mathbf{E}^{\mathrm{T}}(\mu)[\beta]^{-1}\mathbf{E}(\mu)$ vanishes for some μ.

In the contrary case, $\Gamma(\mu_i)$ has two independent column vectors for every eigenvalue μ_i, so that there is exactly one independent eigenvector corresponding to each simple or repeated eigenvalue. Then the material is *normal*.

3. Normal materials

For normal materials the symmetric matrix $\Gamma(\mu)$ has two independent column vectors for every complex number μ. Hence the adjoint matrix $\hat{\Gamma}(\mu)$ does not vanish. Eq. (4) has one independent solution because $\mathbf{a}$ must be orthogonal to the two independent column vectors of $\Gamma(\mu_0)$. Evaluating the following identity at $\mu = \mu_0$

$$\Gamma\hat{\Gamma} = \hat{\Gamma}\Gamma = \Delta\mathbf{I}, \tag{13}$$

one obtains

$$\Gamma(\mu_0)\hat{\Gamma}(\mu_0) = \hat{\Gamma}(\mu_0)\Gamma(\mu_0) = \mathbf{0} \tag{14}$$

Hence the three column vectors of $\hat{\Gamma}(\mu_0)$ are all proportional to $\mathbf{a}$, i.e, the symmetric matrix $\hat{\Gamma}(\mu_0)$ is of rank one. Not all diagonal elements of $\hat{\Gamma}(\mu_0)$ may vanish because otherwise the off-diagonal elements $\hat{\Gamma}_{ij} = \sqrt{(\hat{\Gamma}_{ii}\hat{\Gamma}_{jj})}$ (with no sum on repeated indices) would also vanish and $\hat{\Gamma}(\mu_0)$ would be the null matrix. We choose the column vector of $\hat{\Gamma}$ containing a nonvanishing diagonal element $\gamma_k(\mu)$ as the **a**-vector. Then

$$\hat{\Gamma}(\mu_0) = \mathbf{a}\mathbf{a}^{\mathrm{T}}/\gamma_k(\mu_0) \tag{15}$$

Eq. (7b) gives the associated **b**-vector:

$$\mathbf{b} = -\mathbf{E}'^{\mathrm{T}}(\mu_0)[\beta]^{-1}\mathbf{E}(\mu_0)\mathbf{a} = -(\mathbf{R}^{\mathrm{T}} + \mu_0\mathbf{T})\mathbf{a} \tag{16}$$

Differentiation of Eq. (13) yields

$$\Gamma\hat{\Gamma}' + \Gamma'\hat{\Gamma} = \Delta'\mathbf{I} \tag{17}$$

Premultiplying by $\hat{\Gamma}$, evaluating the result at $\mu = \mu_0$, and using Eqs. (14) and (15), one obtains,

$$\hat{\Gamma}(\mu_0)\Gamma'(\mu_0)\hat{\Gamma}(\mu_0) = \Delta'(\mu_0)\hat{\Gamma}(\mu_0) = \{\Delta'(\mu_0)/\gamma_k\}\mathbf{a}\mathbf{a}^{\mathrm{T}} \tag{18}$$

Since $\mathbf{a}$ is chosen to be the kth column of $\hat{\Gamma}(\mu_0)$, the kth diagonal element of the preceding matrix identity has the expression

$$\mathbf{a}^{\mathrm{T}}\Gamma'(\mu_0)\mathbf{a} = (1/\gamma_k)\gamma_k^2\Delta'(\mu_0) = \gamma_k\Delta'(\mu_0). \tag{19}$$

Consequently

$$\mathbf{b}^{\mathrm{T}}\mathbf{a} + \mathbf{a}^{\mathrm{T}}\mathbf{b} = -\mathbf{a}^{\mathrm{T}}\{\mathbf{R} + \mu_0\mathbf{T} + \mathbf{R}^{\mathrm{T}} + \mu_0\mathbf{T}\}\mathbf{a} = -\mathbf{a}^{\mathrm{T}}\Gamma'(\mu_0)\mathbf{a} = -\gamma_k\Delta'(\mu_0) \tag{20}$$

Hence, the eigenvectors corresponding to distinct eigenvalues μ_i and μ_j satisfy the orthogonality relation

$$\mathbf{b}^{(i)\mathrm{T}}\mathbf{a}^{(j)} + \mathbf{a}^{(i)\mathrm{T}}\mathbf{b}^{(j)} = -\mathbf{a}^{(i)\mathrm{T}}\left(\mathbf{R}^{\mathrm{T}} + \mu_j\mathbf{T} + \mathbf{R} + \mu_j\mathbf{T}\right)\mathbf{a}^{(j)} = -\mathbf{a}^{(i)\mathrm{T}}\{(\Gamma(\mu_j) - \Gamma(\mu_i))/(\mu_j - \mu_i)\}\mathbf{a}^{(j)} = 0 \tag{21}$$

In particular, $\mathbf{b}^{(i)\mathrm{T}}\bar{\mathbf{a}}^{(j)} + \mathbf{a}^{(i)\mathrm{T}}\bar{\mathbf{b}}^{(j)} = \mathbf{0}$, where $\bar{\mathbf{a}}^{(j)}$ and $\bar{\mathbf{b}}^{(j)}$ are, respectively, the complex conjugates of $\mathbf{a}^{(i)}$ and $\mathbf{b}^{(i)}$.

We now examine the cases of simple, double and triple eigenvalues separately.

3.1. N-simple materials (the SP group)

The equation $\Delta(\mu) = 0$ has three distinct roots μ_s ($s = 1, 2, 3$) with positive imaginary parts. As shown before, to each μ_s is associated a unique pair of eigenvectors $\{\mathbf{a}^{(s)}, \mathbf{b}^{(s)}\}$, where $\mathbf{a}^{(s)}$ is a column vector of $\hat{\Gamma}(\mu_s)$ containing a non-zero diagonal element and where $\mathbf{b}^{(s)} = -(\mathbf{R}^{\mathrm{T}} + \mu_s\mathbf{T})\mathbf{a}^{(s)}$. Let $\mathbf{A} \equiv \{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}\}$ and $\mathbf{B} \equiv \{\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \mathbf{b}^{(3)}\}$. Then Eqs. (20) and (21) yield

$$\mathbf{B}^{\mathrm{T}}\bar{\mathbf{A}} + \mathbf{A}^{\mathrm{T}}\bar{\mathbf{B}} = \mathbf{0}, \tag{22}$$

$$\Omega \equiv \mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} -\gamma_k(\mu_1)\Delta'(\mu_1) & 0 & 0 \\ 0 & -\gamma_k(\mu_2)\Delta'(\mu_2) & 0 \\ 0 & 0 & -\gamma_k(\mu_3)\Delta'(\mu_3) \end{bmatrix} \tag{23}$$

Consequently, the Barnett–Lothe tensors

$$\mathbf{H} = -2i\mathbf{A}\Omega^{-1}\mathbf{A}^{\mathrm{T}}, \qquad \mathbf{S} = -i\left(2\mathbf{A}\Omega^{-1}\mathbf{B}^{\mathrm{T}} - \mathbf{I}\right), \qquad \mathbf{L} = 2i\mathbf{B}\Omega^{-1}\mathbf{B}^{\mathrm{T}} \tag{24}$$

have the expressions

$$\mathbf{H} = 2i\sum\left(1/\Delta'(\mu_s)\right)\hat{\Gamma}(\mu_s),$$

$$\mathbf{L} = -2i\sum\left(1/\Delta'(\mu_s)\right)\left(\mathbf{R}^{\mathrm{T}} + \mu_s\mathbf{T}\right)\hat{\Gamma}(\mu_s)(\mathbf{R} + \mu_s\mathbf{T}),$$

$$\mathbf{S} = -2i\sum\left(1/\Delta'(\mu_s)\right)\hat{\Gamma}(\mu_s)(\mathbf{R} + \mu_s\mathbf{T}) + i\mathbf{I}. \tag{25}$$

The first expression of Eq. (25) was obtained by Ting and Lee (1996) using a different derivation.

3.2. N-double materials (the D1 group)

$\Delta(\mu) = 0$ has one simple root $\hat{\mu}$ and one double root μ_0 such that $\hat{\Gamma}(\mu_0)$ is not the null matrix. Let $\hat{\mathbf{a}}$ denote a nonvanishing column vector of $\hat{\Gamma}(\hat{\mu})$ and let $\mathbf{a}$ denote a column vector of $\hat{\Gamma}(\mu_0)$ containing a nonzero diagonal element γ_k. Each is the *unique* independent eigenvector associated with the respective eigenvalue. Hence

$$\Gamma\left(\hat{\mu}\right)\hat{\mathbf{a}} = \mathbf{0}, \qquad \Gamma(\mu_0)\mathbf{a} = \mathbf{0} \tag{26a,b}$$

The two eigensolutions associated with these eigenvectors must be supplemented by an additional solution, given by

$$\mathbf{u} = \mathbf{a}^* f(x + \mu_0 y) + \mathbf{a} y f'(x + \mu_0 y), \qquad \mathbf{q} = \mathbf{b}^* f(x + \mu_0 y) + \mathbf{b} y f'(x + \mu_0 y) \tag{27}$$

where $\mathbf{a}^*$ and $\mathbf{b}^*$ must satisfy (see Section 4 of Part I)

$$\mathbf{E}(\mu_0)\mathbf{a}^* + \mathbf{E}'(\mu_0)\mathbf{a} = [\beta]\mathbf{P}(\mu_0)\begin{Bmatrix} b_2^* \\ b_3^* \end{Bmatrix} + [\beta]\mathbf{P}'(\mu_0)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} \tag{28}$$

Premultiplying the last equation by $\mathbf{E}^{\mathrm{T}}(\mu_0)[\beta]^{-1}$, using $\mathbf{P}' = \mathbf{YJ}$, $\mathbf{E}^{\mathrm{T}}\mathbf{Y} = \mathbf{I}$, $\mathbf{E}^{\mathrm{T}}\mathbf{P} = \mathbf{0}$ and Eq. (7b), one obtains the following equation governing $\mathbf{a}^*$:

$$\Gamma(\mu_0)\mathbf{a}^* + \Gamma'(\mu_0)\mathbf{a} = \mathbf{0} \tag{29}$$

For a double root μ_0, Eq. (17) reduces to

$$\Gamma(\mu_0)\hat{\Gamma}'(\mu_0) + \Gamma'(\mu_0)\hat{\Gamma}(\mu_0) = \mathbf{0} \tag{30}$$

Since $\mathbf{a}$ is a nonvanishing column vector of $\hat{\Gamma}(\mu_0)$, the last equation shows that Eq. (29) is satisfied by choosing $\mathbf{a}^*$ to be the corresponding column vector of $\hat{\Gamma}'(\mu_0)$, so that $\mathbf{a}^* = \mathrm{d}\mathbf{a}/\mathrm{d}\mu$. Then the vector $\mathbf{b}^*$ may be determined from Eq. (28), which, when pre-multiplied by $(\mathrm{d}\mathbf{E}^{\mathrm{T}}/\mathrm{d}\mu)[\beta]^{-1}$, gives the following expression evaluated at $\mu = \mu_0$:

$$\mathbf{b}^* = -(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\mathbf{a}^* - \mathbf{Ta} = \frac{\mathrm{d}}{\mathrm{d}\mu}\left[-(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\mathbf{a}\right] \tag{31}$$

Thus the generalized eigenvectors $\mathbf{a}^*$ and $\mathbf{b}^*$ can be obtained by the derivative rule.

Eq. (17) yields $\mathbf{a}^{\mathrm{T}}\ \mathbf{b}\ +\ \mathbf{b}^{\mathrm{T}}\ \mathbf{a}\ =\ 0$. The higher order derivatives of Eq. (13)

$$\Gamma\hat{\Gamma}'' + 2\Gamma'\hat{\Gamma}' + 2\mathbf{T}\hat{\Gamma} = \Delta''\mathbf{I}, \qquad \Gamma\hat{\Gamma}''' + 3\Gamma'\hat{\Gamma}'' + 3\mathbf{T}\hat{\Gamma}' = \Delta'''\mathbf{I} \tag{32a,b}$$

may be used to obtain

$$\mathbf{a}^{*\mathrm{T}}\mathbf{b} + \mathbf{b}^{*\mathrm{T}}\mathbf{a} = -\gamma_k(\mu_0)\Delta''(\mu_0)/2,$$

$$\mathbf{a}^{*\mathrm{T}}\mathbf{b}^* + \mathbf{b}^{*\mathrm{T}}\mathbf{a}^* = -\gamma_k(\mu_0)\Delta'''(\mu_0)/6 - \gamma_k'(\mu_0)\Delta''(\mu_0)/2$$

Since $\gamma_k(\mu_0)$ was defined as a nonvanishing diagonal element of $\hat{\Gamma}(\mu_0)$ and since $\Delta''(\mu_0)\neq 0$ for a double eigenvalue, the first of the preceding two expressions implies that $\mathbf{a}^*$ and $\mathbf{b}^*$ are not null vectors. Let $\mathbf{A} = \{\hat{\mathbf{a}}, \mathbf{a}, \mathbf{a}^*\}$ and $\mathbf{B}$ be the matrix formed of the corresponding $\mathbf{b}$-vectors. One obtains

$$\Omega \equiv \mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} \Delta'(\hat{\mu}) & 0 & 0 \\ 0 & 0 & -\gamma_k(\mu_0)\Delta''(\mu_0)/2 \\ 0 & -\gamma_k(\mu_0)\Delta''(\mu_0)/2 & -\gamma_k(\mu_0)\Delta'''(\mu_0)/6 - \gamma_k'(\mu_0)\Delta''(\mu_0)/2 \end{bmatrix} \tag{33}$$

This expression is identical to Eq. (4.12) of Part I except that l_2, δ and their derivatives in the latter equation are replaced, respectively, by $-\gamma_k$, Δ and the corresponding derivatives. The Barnett–Lothe tensors are

$$\mathbf{H} = 2i\Big[\{1/\Delta'(\hat{\mu})\}\hat{\Gamma}(\hat{\mu}) + (2/\Delta'')\hat{\Gamma}'(\mu_0) + (2/3)(1/\Delta'')'\hat{\Gamma}(\mu_0)\Big],$$

$$\mathbf{L} = -2i\Big[\{1/\Delta'(\hat{\mu})\}(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T})(\hat{\mu}) + (2/\Delta'')\big[(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T})\big]'(\mu_0) + (2/3) \times (1/\Delta'')'(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T})(\mu_0)\Big],$$

$$\mathbf{S} = -2i\Big[\{1/\Delta'(\hat{\mu})\}\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T})(\hat{\mu}) + (2/\Delta'')\big[\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T})\big]'(\mu_0) + (2/3)(1/\Delta'')'\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T})(\mu_0)\Big] + i\mathbf{I} \tag{34}$$

3.3. N-triple materials (extra-degenerate case)

$\Delta(\mu) = 0$ has a triple root μ_0 and $\hat{\Gamma}(\mu_0)$ has one independent column vector. We choose a column vector of $\hat{\Gamma}(\mu_0)$ containing a nonvanishing diagonal element as the eigenvector $\mathbf{a}$, and let the corresponding column vectors of the symmetric matrices $\hat{\Gamma}'(\mu_0)$ and $\hat{\Gamma}''(\mu_0)$ be denoted by $\mathbf{a}^*$ and $\mathbf{a}^{**}$, respectively. For a triple eigenvalue μ_0 one has

$$\Gamma(\mu_0)\hat{\Gamma}''(\mu_0) + 2\Gamma'(\mu_0)\hat{\Gamma}'(\mu_0) + \Gamma''(\mu_0)\hat{\Gamma}(\mu_0) = \Delta''(\mu_0)\mathbf{I} = \mathbf{0}.$$

Hence

$$\Gamma(\mu_0)\mathbf{a}^{**} + 2\Gamma'(\mu_0)\mathbf{a}^* + \Gamma''(\mu_0)\mathbf{a} = 0 \tag{35}$$

Eqs. (29) and (35) ensure that, besides $\mathbf{u} = \mathbf{a}f(z)$ and $\mathbf{q} = \mathbf{b}f(z)$, there is a second solution given by Eq. (27) satisfying the eigenrelation of Eq. (28) and a third solution given by

$$\mathbf{u} = \mathbf{a}^{**}f(x + \mu_0 y) + \mathbf{a}^* 2yf'(x + \mu_0 y) + \mathbf{a}y^2 f''(x + \mu_0 y),$$

$$\mathbf{q} = \mathbf{b}^{**}f(x + \mu_0 y) + \mathbf{b}^* 2yf'(x + \mu_0 y) + \mathbf{b}y^2 f''(x + \mu_0 y), \tag{36}$$

which satisfies the eigenrelation

$$\mathbf{E}(\mu)\mathbf{a}^{**} + 2\mathbf{E}'(\mu)\mathbf{a}^* + \mathbf{E}''(\mu)\mathbf{a} = [\beta]\left(\mathbf{P}(\mu)\begin{Bmatrix} b_2^{**} \\ b_3^{**} \end{Bmatrix} + 2\mathbf{P}'(\mu)\begin{Bmatrix} b_2^* \\ b_3^* \end{Bmatrix} + \mathbf{P}''(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}\right) \tag{37}$$

The vectors $\mathbf{b}$, $\mathbf{b}^*$ and $\mathbf{b}^{**}$ are as defined by Eqs. (16), (31) and

$$\mathbf{b}^{**} = -(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\mathbf{a}^{**} - 2\mathbf{T}\mathbf{a}^* = \frac{\mathrm{d}^2}{\mathrm{d}\mu^2}\big[-(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\mathbf{a}\big], \tag{38}$$

with the right hand expression evaluated at $\mu = \mu_0$. Hence $\mathbf{b}^{**}$ is also obtained by the derivative rule. Let $\mathbf{B} = \{\mathbf{b}, \mathbf{b}^*, \mathbf{b}^{**}\}$ and $\mathbf{A} = \{\mathbf{a}, \mathbf{a}^*, \mathbf{a}^{**}\}$. Then $\Omega \equiv \mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B}$ becomes

$$\Omega = \begin{bmatrix} 0 & 0 & -\gamma_k\Delta'''/3 \\ 0 & -\gamma_k\Delta'''/6 & -\gamma_k\Delta''''/12 - \gamma_k'\Delta'''/3 \\ -\gamma_k\Delta'''/3 & -\gamma_k\Delta''''/12 - \gamma_k'\Delta'''/3 & -\gamma_k\Delta'''''/30 - \gamma_k'\Delta''''/6 - \gamma_k''\Delta'''/3 \end{bmatrix} \tag{39}$$

where γ_k is the nonvanishing diagonal element of $\hat{\Gamma}(\mu_0)$ contained in the column vector $\mathbf{a}$. This expression is identical to Eq. (5.13a) in Part I of this paper except that l_2, δ and their derivatives are replaced, respectively, by $-\gamma_k$, Δ and the corresponding derivatives. The Barnett–Lothe tensors are

$$\mathbf{H} = (6i/\Delta''')\hat{\Gamma}'' + 3i(1/\Delta''')'\hat{\Gamma}' + (6i/19)(1/\Delta''')''\hat{\Gamma},$$

$$\mathbf{L} = -(6i/\Delta''')\left\{(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T})\right\}'' - 3i(1/\Delta''')'\left\{(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T})\right\}' - (6i/19)(1/\Delta''')''(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T}),$$

$$\mathbf{S} = -(6i/\Delta''')\left\{\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T})\right\}'' - 3i(1/\Delta''')'\left\{\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T})\right\}' - (6i/19)(1/\Delta''')''\hat{\Gamma}(\mathbf{R} + \mu\mathbf{T}) + i\mathbf{I}. \tag{40}$$

4. Examples of extra-degenerate materials

Materials possessing a triple eigenvalue (N-triple and A-triple materials) may be characterized by explicit relations among the elements of $[\beta]$, or of $[\beta]^{-1}$. In terms of these elements, the triple eigenvalue $\xi \pm i\eta$ may also be obtained explicitly. By equating the like powers of μ of the following two expressions of $\Delta \equiv |\Gamma(\mu)|$

$$\Delta = |\mathbf{Q} + \mu(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + \mu^2\mathbf{T}| = |\mathbf{T}|(\mu - \xi - i\eta)^3(\mu - \xi + i\eta)^3$$

one obtains six identities involving the elastic constants and the real numbers ξ and η. Through the elimination of ξ and η, four algebraic relations among the fifteen independent elastic constants C_{ij} ($i \leq j$ and $i, j = 1, 2, 4, 5, 6$) are obtained. Such materials are extra-degenerate if they are normal, i.e., if the triple root $\mu_0 = \xi + i\eta$ is *not* a common root of $l_2(\mu) = 0$ and $l_4(\mu) = 0$. An alternative criterion is $\hat{\Gamma}(\mu_0) \neq \mathbf{0}$. Here ξ and η are easily found to be

$$\xi = -(1/3)\mathrm{tr}\left[\mathbf{T}^{-1}(\mathbf{R} + \mathbf{R}^{\mathrm{T}})\right]/|\mathbf{T}|, \qquad \eta = \left[\left\{|\mathbf{Q}|/|\mathbf{T}|\right\}^{1/3} - \xi^2\right]^{1/2} \tag{41}$$

In terms of the elements of $[\beta]$, one has (Ting, 1998)

$$\begin{aligned} \xi &= (1/3)\{\beta_{11}\beta_{45} + \beta_{16}\beta_{55} + \beta_{15}(\beta_{14} + \beta_{56})\}/\left(\beta_{11}\beta_{55} - \beta_{15}^2\right), \\ \eta &= \left[\left\{\left(\beta_{22}\beta_{44} - \beta_{24}^2\right)/\left(\beta_{11}\beta_{55} - \beta_{15}^2\right)\right\}^{1/3} - \xi^2\right]^{1/2} \end{aligned} \tag{42}$$

To obtain a new class of extra-degenerate materials, we let α be any positive number and a be such that $0 < a < 3$ and $a \neq 1$. Consider elastic materials with

$$\beta_{24} = \beta_{16} = \beta_{26} = \beta_{45} = \beta_{14} + \beta_{56} = \beta_{25} + \beta_{46} = 0 \tag{43}$$

and

$$\beta_{55}/\beta_{44} = a/\alpha^2, \qquad (\beta_{66} + 2\beta_{12})/\beta_{22} = (3 - a)/\alpha^2, \tag{44a,b}$$

$$\beta_{11}/\beta_{22} = \left(a^2 - 3a + 3\right)/\alpha^4, \qquad \beta_{15}/\sqrt{\left(\beta_{22}\beta_{44}\right)} = (a-1)^{3/2}/\alpha^3, \tag{44c,d}$$

Then

$$l_2 = \beta_{44}\left\{a(\mu/\alpha)^2 + 1\right\}, \qquad l_3 = \sqrt{\left(\beta_{22}\beta_{44}\right)}(a-1)^{3/2}(\mu/\alpha)^3,$$

$$l_4 = \beta_{22}\left\{\left(a^2 - 3a + 3\right)(\mu/\alpha)^4 + (3-a)(\mu/\alpha)^2 + 1\right\},$$

$$l_2 l_4 - l_3^2 = \beta_{22}\beta_{44}\left\{(\mu/\alpha)^2 + 1\right\}^3,$$

Thus $\mu = \pm i\alpha$ are triple roots and

$$l_2(i\alpha) = -(a-1)\beta_{44}, \qquad l_3(i\alpha) = -i(a-1)^{3/2}\sqrt{\left(\beta_{22}\beta_{44}\right)}, \qquad l_4(i\alpha) = (a-1)^2\beta_{22},$$

do not vanish since $a \neq 1$. The class of materials given above can have a positive definite strain energy if the β_{ij}'s are properly restricted. To demonstrate this contention we consider the subclass of materials that satisfy, in addition, $\beta_{14} = \beta_{25} = 0$. Then $\beta_{24} + \beta_{56} = \beta_{25} + \beta_{46} = 0$ imply that $\beta_{56} = \beta_{46} = 0$. Hence the strain energy density function U_0 has the expression

$$\begin{aligned} 2U_0 &= \beta_{11}\sigma_x^2 + \beta_{22}\sigma_y^2 + 2\beta_{12}\sigma_x\sigma_y + 2\beta_{15}\sigma_x\tau_{xz} + \beta_{55}\tau_{xz}^2 + \beta_{44}\tau_{yz}^2 + \beta_{66}\tau_{xy}^2 \\ &= \left\{\left(\beta_{11}\beta_{55} - \beta_{15}^2\right)/\beta_{55}\right\}\sigma_x^2 + 2\beta_{12}\sigma_x\sigma_y + \beta_{22}\sigma_y^2 + \beta_{55}\left(\sigma_x\beta_{15}/\beta_{55} + \tau_{xz}\right)^2 + \beta_{44}\tau_{yz}^2 + \beta_{66}\tau_{xy}^2. \end{aligned}$$

Eqs. (44a, c and d) yield $(\beta_{11}\beta_{55} - \beta_{15}^2)/\beta_{55} = \beta_{22}/(a\alpha^4)$. Consequently,

$$2U_0 = \beta_{22}\left\{\sigma_x^2/(a\alpha^4) + 2\sigma_x\sigma_y\beta_{12}/\beta_{22} + \sigma_y^2\right\} + \beta_{55}\left(\sigma_x\beta_{15}/\beta_{55} + \tau_{xz}\right)^2 + \beta_{44}\tau_{yz}^2 + \beta_{66}\tau_{xy}^2$$

and U_0 is positive definite if $1/(a\alpha^4) > (\beta_{12}/\beta_{22})^2$. This inequality condition is ensured by Eq. (44b) if the following restriction is made on β_{66}/β_{22}:

$$|\beta_{66}/\beta_{22} - (3-a)/\alpha^2| < 2/\left(\alpha^2\sqrt{a}\right).$$

The subclass of materials satisfying this restriction is certainly not empty since β_{22} and β_{66} are independent elastic constants of anisotropic materials.

The extra-degenerate materials found above are different from the examples given by Ting (1996b), although the two classes share certain similar features. In defining both classes of materials the reference coordinate axes x and y have been chosen so that the triple eigenvalue is purely imaginary. These are intrinsic material axes, not arbitrarily picked directions. In Ting's examples $\beta_{15} = \beta_{24} = 0$. This implies that a uniaxial stress along the x-direction will not produce anti-plane shear strain γ_{xz}, and a uniaxial stress along the y-direction will not produce γ_{yz}. In the present examples, β_{24} vanishes but β_{15} does not. Hence the behavior of the two classes of materials is different.

5. Abnormal materials

Abnormal materials have an eigenvalue μ_0 such that $\Gamma(\mu_0)$ has only one independent column vector. Then the elements of the adjoint matrix, which are 2×2 minors of $\Gamma(\mu_0)$, must all vanish:

$$\hat{\Gamma}(\mu_0) = \mathbf{0}. \tag{45}$$

$$\Gamma(\mu_0)\hat{\Gamma}'(\mu_0) = \Delta'(\mu_0)\mathbf{I}. \tag{46}$$

Since $\Gamma(\mu_0)$ is singular, Eq. (46) implies that $\Delta'(\mu_0) = 0$. Hence μ_0 is a repeated eigenvalue and

$$\Gamma(\mu_0)\hat{\Gamma}'(\mu_0) = \mathbf{0}. \tag{47}$$

By eliminating the terms of the third and fourth power in μ from the six quartic equations $\hat{\Gamma}_{ij}(\mu) = 0$ (i, j = 1, 2, 3), one obtains quadratic equations that can be solved for the repeated eigenvalue μ_0. Once μ_0 is determined, the remaining eigenvalue $\hat{\mu}$, if different from μ_0, may be obtained from the quadratic equation $(\mu - \mu_0)^{-2}(\mu - \bar{\mu})^{-2}\Delta(\mu) = 0$. However, simpler explicit expressions of the eigenvalues of abnormal materials may be given in terms of the anisotropic elastic compliances β_{ij} (Ting, 1999).

We now show that $\hat{\Gamma}'(\mu_0)$ is not the null matrix. Suppose the contrary, then μ_0 would be a double root of all elements of $\hat{\Gamma}(\mu)$, i.e.,

$$\hat{\Gamma}(\mu) = (\mu - \mu_0)^2(\mu - \bar{\mu}_0)^2\hat{\mathbf{T}},$$

where $\hat{\mathbf{T}} \equiv |\mathbf{T}|\mathbf{T}^{-1}$ denotes the adjoint of the constant symmetric matrix $\mathbf{T}$. Then Eq. (13) yields

$$\Gamma(\mu) = (\mu - \mu_0)^{-2}(\mu - \bar{\mu}_0)^{-2}\{\Delta(\mu)/|\mathbf{T}|\}\mathbf{T}.$$

Since $\Gamma(\mu_0)$ is singular but $\mathbf{T}$ is not, The last identity implies that μ_0 must be a triple root of $\Delta(\mu) = 0$ and therefore $\Delta(\mu) = (\mu - \mu_0)^3(\mu - \bar{\mu}_0)^3|\mathbf{T}|$. Then

$$\Gamma(\mu) = (\mu - \mu_0)(\mu - \bar{\mu}_0)\mathbf{T},$$

so that $\Gamma(\mu_0) = \mathbf{0}$. As shown in Section 2, this may happen only if the elasticity matrix $[\beta]^{-1}$ is singular. Otherwise the supposition $\hat{\Gamma}'(\mu_0) = \mathbf{0}$ cannot be valid.

Eq. (47) shows that the column vectors of $\hat{\Gamma}'(\mu_0)$ are solutions of Eq. (4). There are at most two independent column vectors because each vector must be orthogonal to the unique independent row vector of $\Gamma(\mu_0)$. The independent column vectors of $\hat{\Gamma}'(\mu_0)$ will be chosen as independent **a**-vectors. If $\hat{\Gamma}'(\mu_0)$ has only one independent column vector $\mathbf{a}^{(1)}$, then a second **a**-vector may be chosen as $\mathbf{a}^{(2)} = \mathbf{v} \times \mathbf{a}^{(1)}$, where $\mathbf{v}$ is a nonvanishing column of $\Gamma(\mu_0)$.

For A-double materials the two independent **a**-vectors associated with the double eigenvalue μ_0 are supplemented by another **a**-vector associated with a simple eigenvalue $\hat{\mu}$. The latter may be chosen as a nonvanishing column vector of $\hat{\Gamma}(\hat{\mu})$. For each **a**-vector, Eq. (16) yields the corresponding **b**-vector.

For A-triple materials the two independent **a**-vectors associated with the triple eigenvalue μ_0 are supplemented by a generalized eigenvector $\mathbf{a}^*$. Since $\Delta''(\mu_0) = 0$, Eq. (32a) reduces to

$$\Gamma(\mu_0)\hat{\Gamma}''(\mu_0) + 2\Gamma'(\mu_0)\hat{\Gamma}'(\mu_0) = \mathbf{0}. \tag{48}$$

At least one of the two independent **a**-vectors is a column vector of $\hat{\Gamma}'(\mu_0)$. Hence Eq. (29) is satisfied if $\mathbf{a}^*$ is taken to be (1/2) times the corresponding column vector of $\hat{\Gamma}''(\mu_0)$.

The Barnett–Lothe tensors of A-double and A-triple materials are given by the expressions for N-Double and N-Triple materials, respectively, where the terms containing $\hat{\Gamma}(\mu_0)$ vanish.

6. Identities involving the Barnett–Lothe tensors

We have shown that for every type of anisotropic materials (nondegenerate, degenerate or extra-degenerate; normal or abnormal) a set of three independent eigenvectors or generalized eigenvectors may be explicitly found in terms of the eigenvalues and the elements of $[\beta]$, or of $[\beta]^{-1}$. Let $\mathbf{A}=\{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}\}$ and $\mathbf{B}=\{\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \mathbf{b}^{(3)}\}$. Then the matrix $\Omega \equiv \mathbf{B}^{\mathrm{T}}\mathbf{A}+\mathbf{A}^{\mathrm{T}}\mathbf{B}$ is nonsingular, and the orthogonality relations of Eqs. (22) and (23) are satisfied. As shown in Part I, these relations imply that $\mathbf{L}$, $\mathbf{H}$ and $\mathbf{S}$ as defined by Eq. (24) are real matrices, and $\mathbf{L}$ and $\mathbf{H}$ are symmetric and non-singular. The three tensors may also be expressed as follows

$$\mathbf{N}^* \equiv \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} \begin{bmatrix} -i\Omega^{-1} & \mathbf{0} \\ \mathbf{0} & i\bar{\Omega}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} \tag{49a}$$

$$\begin{bmatrix} \mathbf{H} & \mathbf{S} \\ \mathbf{S}^{\mathrm{T}} & -\mathbf{L} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} \begin{bmatrix} -i\Omega^{-1} & \mathbf{0} \\ \mathbf{0} & i\bar{\Omega}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{\mathrm{T}} & \mathbf{B}^{\mathrm{T}} \\ \bar{\mathbf{A}}^{\mathrm{T}} & \bar{\mathbf{B}}^{\mathrm{T}} \end{bmatrix} \tag{49b}$$

Multiplication of the preceding two matrices and use of Eqs. (22) and (23) yields

$$\begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{H} & \mathbf{S} \\ \mathbf{S}^{\mathrm{T}} & -\mathbf{L} \end{bmatrix} = \begin{bmatrix} \mathbf{SH}+\mathbf{HS}^{\mathrm{T}} & -\mathbf{HL}+\mathbf{SS} \\ -\mathbf{LH}+\mathbf{S}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} & -\mathbf{LS}-\mathbf{S}^{\mathrm{T}}\mathbf{L} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}.$$

Hence

$$\mathbf{HL}-\mathbf{SS}=\mathbf{I}=\mathbf{LH}-\mathbf{S}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}, \tag{50}$$

and $\mathbf{SH}$ and $\mathbf{LS}$ are skew-symmetric. Then the matrices

$$\mathbf{H}^{-1}\mathbf{S}=\mathbf{H}^{-1}(\mathbf{SH})\mathbf{H}^{-1}, \qquad \mathbf{SL}^{-1}=\mathbf{L}^{-1}(\mathbf{LS})\mathbf{L}^{-1}$$

are also skew-symmetric. Furthermore, Eq. (24) yields

$$\mathbf{AB}^{-1}\mathbf{L}=2i\mathbf{A}\Omega^{-1}\mathbf{B}^{\mathrm{T}}=-\mathbf{S}+i\mathbf{I}, \quad \mathbf{BA}^{-1}\mathbf{H}=-2i\mathbf{B}\Omega^{-1}\mathbf{A}^{\mathrm{T}}=\mathbf{S}^{\mathrm{T}}-i\mathbf{I}.$$

Hence,

$$\mathbf{AB}^{-1}=-\mathbf{SL}^{-1}+i\mathbf{L}^{-1}, \quad \mathbf{BA}^{-1}=\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1}-i\mathbf{H}^{-1}. \tag{51}$$

Notice that the real parts of $\mathbf{AB}^{-1}$ and $\mathbf{BA}^{-1}$ are skew-symmetric and the imaginary parts are symmetric. Eq. (51) also implies that $\mathbf{I} + i\mathbf{S}$ is a nonsingular matrix.

For nondegenerate materials, the preceding relations among the matrices $\mathbf{L}$, $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{AB}^{-1}$ are well known. A proof for degenerate (but not extra-degenerate) materials was given by Ting and Hwu (1988). Using the general relations of Eqs. (22), (24) and $\Omega \equiv \mathbf{B}^{\mathrm{T}}\mathbf{A}+\mathbf{A}^{\mathrm{T}}\mathbf{B}$ for unnormalized eigen vectors and generalized eigenvectors. the present proof establishes the validity of Eqs. (50), (51) and the skew-symmetry of $\mathbf{SH}$ and $\mathbf{LS}$ for all types of anisotropic materials.

Eq. (50) implies that any one of the three matrices $\mathbf{L}$, $\mathbf{H}$ and $\mathbf{SS}$ is determined by the other two. However, $\mathbf{S}$ is determined by $\mathbf{L}$ and $\mathbf{H}$ only up to a scalar factor ± 1. To calculate $\pm\mathbf{S}$ in terms of $\mathbf{L}$ and $\mathbf{H}$, we first notice that $\mathbf{S}=(\mathbf{SL}^{-1})\mathbf{L}$ is the product of a skew-symmetric matrix and a symmetric matrix. Hence $-\mathbf{S}^{\mathrm{T}}=\mathbf{L}(\mathbf{SL}^{-1})$. It follows that $\mathbf{S}$ and $-\mathbf{S}^{\mathrm{T}}$ have the same trace and determinant, and hence both invariants must vanish. Then the characteristic equation of $\mathbf{S}$ has the form $\lambda^3+s^2\lambda=0$ where (Ting, 1996a)

$$s^2 = -\mathrm{tr}(\mathbf{SS})/2 = 3/2 - \mathrm{tr}(\mathbf{HL})/2. \tag{52}$$

Since $\mathbf{S}$ satisfies its own characteristic equation, one has $\mathbf{S}^3 = -s^2\mathbf{S}$. Eq. (50) then yields $\mathbf{S} = \mathbf{SHL} - \mathbf{S}^3 = \mathbf{SHL} + s^2\mathbf{S}$. Hence the skew-symmetric matrices $\mathbf{SH}$ and $\mathbf{SL}^{-1}$ satisfy

$$\mathbf{SH}\left\{(1 - s^2)\mathbf{H}^{-1} - \mathbf{L}\right\} = \mathbf{0}, \quad \mathbf{SH} = (1 - s^2)\mathbf{SL}^{-1}. \tag{53}$$

Similarly, the following equations are satisfied by $\mathbf{LS}$ and $\mathbf{H}^{-1}\mathbf{S}$

$$\left\{(1 - s^2)\mathbf{L}^{-1} - \mathbf{H}\right\}\mathbf{LS} = \mathbf{0}, \qquad \mathbf{LS} = (1 - s^2)\mathbf{H}^{-1}\mathbf{S}. \tag{54a,b}$$

It is an algebraic theorem that if $\mathbf{ZW} = \mathbf{0}$ for a 3×3 symmetric matrix $\mathbf{Z}$ and a nonvanishing 3×3 skew-symmetric matrix $\mathbf{W}$, then all three column vectors of $\mathbf{Z}$ must be proportional to the axial vector of $\mathbf{W}$ (the axial vector is defined by $\mathbf{w} = \{W_{23}, -W_{13}, W_{12}\}^{\mathrm{T}}$ in terms of the nonvanishing elements of $\mathbf{W}$), i.e., $\mathbf{Z}$ is a scalar multiple of

$$\mathbf{ww}^{\mathrm{T}} = \mathbf{WW} - (1/2)\mathrm{tr}(\mathbf{WW})\mathbf{I}.$$

Using this theorem in conjunction with Eqs. (52) and (54a, b), we conclude that any nonvanishing column vector $\mathbf{v} = \{v_1, v_2, v_3\}^{\mathrm{T}}$ of the symmetric matrix $\{\mathrm{tr}(\mathbf{HL})/2 - 1/2\}\mathbf{L}^{-1} - \mathbf{H}$ is proportional to the axial vector of the skew-symmetric matrix $\mathbf{H}^{-1}$ $\mathbf{S}$. Hence

$$\mathbf{H}^{-1}\mathbf{S} = \rho\mathbf{V} \tag{55}$$

for some real scalar ρ, where

$$\mathbf{V} = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} \tag{56}$$

Then $\mathbf{S} = \rho\mathbf{HV}$ and $\mathbf{SS} = \mathbf{HL} - \mathbf{I} = \rho^2\mathbf{HVHV}$. Consequently

$$\mathbf{L} - \mathbf{H}^{-1} = \rho^2\mathbf{VHV}, \tag{57}$$

so that

$$\rho^2 = \left\{\mathrm{tr}(\mathbf{L}) - \mathrm{tr}(\mathbf{H}^{-1})\right\}/\mathrm{tr}(\mathbf{VHV}) \tag{58}$$

Then $\mathbf{S} = \rho\mathbf{HV}$ is completely determined except for a scalar factor ± 1 involved in the definition of $\mathbf{V}$. This scalar factor is not determined by $\mathbf{L}$ and $\mathbf{H}$.

Besides the preceding relations among $\mathbf{L}$, $\mathbf{H}$ and $\mathbf{S}$, additional relations may be found that involve also the elasticity matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$. These relations may be used to simplify the explicit expressions of the Barnett–Lothe tensors.

Using Eq. (16) and $\mathbf{b} = \mathbf{BA}^{-1}\mathbf{a}$, one obtains

$$\left(\mathbf{BA}^{-1} + \mathbf{R}^{\mathrm{T}} + \mu\mathbf{T}\right)\mathbf{a} = \mathbf{0}. \tag{59}$$

Substitution into $\Gamma(\mu)\mathbf{a} = \mathbf{0}$ yields

$$\left\{\mathbf{Q} + \mu\left(\mathbf{R} - \mathbf{BA}^{-1}\right)\right\}\mathbf{a} = \mathbf{0}. \tag{60}$$

Premultiplying Eq. (59) by $(\mathbf{R} - \mathbf{BA}^{-1})\mathbf{T}^{-1}$ and substituting the result into Eq. (60), one obtains the following equation that does not contain the eigenvalue μ:

$$\{(\mathbf{R}-\mathbf{B}\mathbf{A}^{-1})\mathbf{T}^{-1}(\mathbf{R}^{\mathrm{T}}+\mathbf{B}\mathbf{A}^{-1})-\mathbf{Q}\}\mathbf{a}=\mathbf{0}. \tag{61}$$

In the case of generalized eigenvectors $\mathbf{a}^*$ and $\mathbf{a}^{**}$, one may use Eqs. (31) and (38) to obtain

$$(\mathbf{R}^{\mathrm{T}}+\mu\mathbf{T}+\mathbf{B}\mathbf{A}^{-1})\mathbf{a}^*=-\mathbf{T}\mathbf{a}, \tag{62a}$$

$$(\mathbf{R}^{\mathrm{T}}+\mu\mathbf{T}+\mathbf{B}\mathbf{A}^{-1})\mathbf{a}^{**}=-2\mathbf{T}\mathbf{a}^*. \tag{62b}$$

Substitution into Eqs. (29) and (35) yields

$$\{\mathbf{Q}+\mu(\mathbf{R}-\mathbf{B}\mathbf{A}^{-1})\}\mathbf{a}^*+(\mathbf{R}-\mathbf{B}\mathbf{A}^{-1})\mathbf{a}=\mathbf{0}, \tag{63a}$$

$$\{\mathbf{Q}+\mu(\mathbf{R}-\mathbf{B}\mathbf{A}^{-1})\}\mathbf{a}^{**}+2(\mathbf{R}-\mathbf{B}\mathbf{A}^{-1})\mathbf{a}^*=\mathbf{0}. \tag{63b}$$

Premultiplying Eqs. (62a) and (62b) by $(\mathbf{R}-\mathbf{B}\mathbf{A}^{-1})\mathbf{T}^{-1}$ and combining the results with Eqs. (63a) and (63b), one finds that Eq. (61) is also satisfied by the generalized eigenvectors $\mathbf{a}^*$ and $\mathbf{a}^{**}$. Thus, there are always three independent vectors satisfying Eq. (61). Consequently,

$$(\mathbf{R}-\mathbf{B}\mathbf{A}^{-1})\mathbf{T}^{-1}(\mathbf{R}^{\mathrm{T}}+\mathbf{B}\mathbf{A}^{-1})-\mathbf{Q}=\mathbf{0}. \tag{64}$$

Using Eq. (51), the last equation may be separated into real and imaginary parts:

$$(\mathbf{R}-\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1})\mathbf{T}^{-1}(\mathbf{R}^{\mathrm{T}}+\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1})+\mathbf{H}^{-1}\mathbf{T}^{-1}\mathbf{H}^{-1}=\mathbf{Q}. \tag{65}$$

$$\mathbf{H}^{-1}\mathbf{T}^{-1}(\mathbf{R}^{\mathrm{T}}+\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1})=(\mathbf{R}-\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1})\mathbf{T}^{-1}\mathbf{H}^{-1}. \tag{66}$$

Eq. (66), when premultiplied by **TH** and postmultiplied by **HT**, becomes

$$\mathbf{R}^{\mathrm{T}}\mathbf{H}\mathbf{T}+\mathbf{S}^{\mathrm{T}}\mathbf{T}=\mathbf{T}\mathbf{H}\mathbf{R}+\mathbf{T}\mathbf{S} \tag{67}$$

The last two equations imply the symmetry of the matrices $\mathbf{H}^{-1}\mathbf{T}^{-1}$ $(\mathbf{R}^{\mathrm{T}}+\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1})$ and $\mathbf{T}(\mathbf{H}\mathbf{R}+\mathbf{S})$. Substituting Eq. (66) into Eq. (65), postmultiplying the result by **HT**, and using Eq. (50) and the skew-symmetry of **SH**, one obtains an expression of **L** depending linearly on **H** and **S**:

$$\mathbf{L}=\mathbf{Q}\mathbf{H}\mathbf{T}-\mathbf{R}\mathbf{H}\mathbf{R}-\mathbf{R}\mathbf{S}+\mathbf{S}^{\mathrm{T}}\mathbf{R}. \tag{68}$$

As mentioned before, **S** is determined by **L** and **H** except for a multiplicative factor ± 1. This factor may be determined by using either one of the last two equations. An alternative proof of Eq. (68) may be given on a case by case basis through direct substitution of Eqs. (25), (34) and (40) into Eq. (68) and using Eqs. (13), (17) and (32a) in conjunction with the vanishing of Δ, Δ' and Δ'' for simple, double and triple roots.

Using a different combination of Eqs. (59) and (60), one may obtain, through an analogous derivation, the following result instead of Eq. (64):

$$(\mathbf{R}^{\mathrm{T}}+\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1}-\mathbf{H}^{-1})\mathbf{Q}^{-1}(\mathbf{R}-\mathbf{S}^{\mathrm{T}}\mathbf{H}^{-1}+\mathbf{H}^{-1})=\mathbf{T}. \tag{69}$$

The real part of Eq. (69) again yields Eq. (68). The imaginary part implies symmetry of $\mathbf{Q}(\mathbf{H}\mathbf{R}^{\mathrm{T}}-\mathbf{S})$.

The eigenvalues and the Barnett–Lothe tensors are determined by the three elasticity matrices **Q**, **R** and **T**. When the Barnett–Lothe tensors are known, Eq. (66) gives the elasticity matrix **Q** in terms of **R** and **T** while Eq. (69) gives **T** in terms of **R** and **Q**.

7. Equivalence of the dual formalisms

A concise and rigorous proof of the equivalence of the two formalisms and, in particular, of the two characteristic equations $\delta(\mu) = 0$ and $\Delta(\mu) = 0$, may be given based on the eigenrelations of Eqs. (3), (28) and (37). We define

$$\mathbf{D} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{70}$$

Then, using Eqs. (11), (2a,b), (8a,b) and (10) and the identity $\mathbf{P}^{\mathrm{T}}\mathbf{E} = 0$, one obtains

$$\begin{aligned} \mathbf{M}\Big(\mathbf{D}\mathbf{E}'^{\,\mathrm{T}}[\beta]^{-1}\mathbf{E}\mathbf{a}\Big) &= \mathbf{P}^{\mathrm{T}}[\beta]\mathbf{P}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}[\beta]^{-1}\mathbf{E}\mathbf{a} = \mathbf{P}^{\mathrm{T}}[\beta]\big(-\mathbf{I} + \mathbf{Y}\mathbf{E}^{\mathrm{T}}\big)[\beta]^{-1}\mathbf{E}\mathbf{a} \\ &= \mathbf{P}^{\mathrm{T}}[\beta]\mathbf{Y}\Gamma\mathbf{a}. \end{aligned} \tag{71}$$

If $\Gamma(\mu)\mathbf{a} = \mathbf{0}$ where μ is a root of $\Delta(\mu) = 0$ and $\mathbf{a}$ is a nontrivial vector, then the last equation implies that $\mathbf{b} = -(\mathbf{E}'^{\,\mathrm{T}}[\beta]^{-1}\mathbf{E}\mathbf{a})$ is a solution of $\mathbf{M}(\mu)\mathbf{D}\mathbf{b} = \mathbf{0}$. The latter vector cannot be the null vector because if it were, then

$$\mathbf{Y}^{\mathrm{T}}[\beta]\mathbf{P}\mathbf{D}\mathbf{b} = -\mathbf{Y}^{\mathrm{T}}[\beta]\mathbf{P}\mathbf{D}\mathbf{E}'^{\,\mathrm{T}}[\beta]^{-1}\mathbf{E}\mathbf{a} = \mathbf{Y}^{\mathrm{T}}[\beta](\mathbf{I} - \mathbf{Y}\mathbf{E}^{\mathrm{T}})[\beta]^{-1}\mathbf{E}\mathbf{a} = \mathbf{Y}^{\mathrm{T}}\mathbf{E}\mathbf{a} - \mathbf{Y}^{\mathrm{T}}[\beta]\mathbf{Y}\Gamma\mathbf{a} = \mathbf{a}$$

would also be the null vector, contradicting the premise that $\mathbf{a}$ is a nontrivial solution of $\Gamma(\mu)\mathbf{a} = \mathbf{0}$. The preceding derivation shows that μ must also be a root of the equation $\delta(\mu) = 0$.

If μ is a repeated root of $\Delta(\mu) = 0$ with more than one independent **a**-vectors, then the preceding proof also implies that there is an equal number of independent **b**-vectors. In the degenerate cases there may be one or two generalized **a**-vectors associated with μ, in addition to one or two independent eigenvectors. For normal materials one of the independent eigenvectors **a** may be chosen as a non-vanishing column vector of $\hat{\Gamma}(\mu)$ so that $\Gamma(\mu)\mathbf{a} = \Delta(\mu)\mathbf{e}$; for abnormal materials **a** may be chosen as a non-vanishing column vector of $\hat{\Gamma}'(\mu)$ so that $\Gamma(\mu)\mathbf{a} = \Delta'(\mu)\mathbf{e}$ (in both cases **e** denotes the corresponding column vector of the 3×3 identity matrix). Then, for the degenerate case (N-double and A-triple materials), $\Gamma\mathbf{a}$ and $(\mathrm{d}/\mathrm{d}\mu)(\Gamma\mathbf{a})$ both vanish when evaluated at the repeated root of $\Delta(\mu) = 0$; for the extra-degenerate case (A-triple materials), $\Gamma\mathbf{a}$ and its first two derivatives vanish when evaluated at the triple root of $\Delta(\mu) = 0$. By differentiating Eq. (71) with respect to μ once or twice, and using Eqs. (29) and (35), one obtains

$$\mathbf{M}\mathbf{D}\mathbf{b}^* + \mathbf{M}'\mathbf{D}\mathbf{b} = \mathbf{0}, \qquad \mathbf{M}\mathbf{D}\mathbf{b}^{**} + 2\mathbf{M}'\mathbf{D}\mathbf{b}^* + \mathbf{M}''\mathbf{D}\mathbf{b} = \mathbf{0},$$

where

$$\mathbf{b} = -\mathbf{E}'^{\,\mathrm{T}}[\beta]^{-1}\mathbf{E}\mathbf{a}, \qquad \mathbf{b}^* = \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}\mu}, \qquad \boldsymbol{b}^{**} = \frac{\mathrm{d}^2\mathbf{b}}{\mathrm{d}\mu^2}.$$

Therefore, there are an equal number of generalized **b**-vectors associated with the eigenvalue μ.

An analogous proof may be given for the converse statement, which starts from the assumption that μ is a root of $\delta(\mu) = 0$ with a nontrivial vector **b** satisfying $\mathbf{M}(\mu)\mathbf{D}\mathbf{b} = 0$, and leads to the conclusion that $\mathbf{Y}^{\mathrm{T}}[\beta]\mathbf{P}\mathbf{D}\mathbf{b}$ is a nontrivial solution of $\Gamma(\mu)\mathbf{a} = 0$. This proof, and its extension to the degenerate and extra-degenerate cases, are omitted for the sake of brevity. The two conclusions taken together imply that μ is a root of $\delta(\mu) = 0$ if and only if it is a root of $\Delta(\mu) = 0$, and that the eigenvectors and generalized eigenvectors in one formalism yield directly those of the other formalism through Eqs. (16),

(31) and (38) or through the converse relations $\mathbf{a} = \mathbf{Y}^{\mathrm{T}}[\beta]\mathbf{PDb}$, $\mathbf{a}^* = (\mathrm{d}/\mathrm{d}\mu)(\mathbf{Y}^{\mathrm{T}}[\beta]\mathbf{PDb})$ and $\mathbf{a}^{**} = (\mathrm{d}^2/\mathrm{d}\mu^2)(\mathbf{Y}^{\mathrm{T}}[\beta]\mathbf{PDb})$.

We note that a direct if somewhat lengthy algebraic proof of the equivalence of the two characteristic equations $\delta(\mu) = 0$ and $\Delta(\mu) = 0$ (but not the equivalence of the eigenspaces of the two formalisms) was given recently by Barnett and Kirchner (1997).

8. The six-dimensional eigenmatrix N

The derivations based on the two formalisms, one in terms of the elements of $[\beta]$ and the other in terms of $[\beta]^{-1}$, lead to the same classification of anisotropic materials into five distinctive types depending on the multiplicity of eigenvalues and, if there is a multiple eigenvalue μ_0, whether or not the adjoint matrix of $\mathbf{M}(\mu_0)$ or that of $\Gamma(\mu_0)$ vanishes. For each type of material, explicit expressions are obtained in both formalisms for the eigenvectors and the associated eigensolutions of the displacements and the stresses. The generalized eigenvectors and the corresponding eigensolutions are given by the derivative rule. Depending on the multiplicity of eigenvalues, various explicit expressions are also given for the Barnett–Lothe tensors in both formalisms. Except for the trivial numerical task of finding the roots of the characteristic equation (even this can be avoided for abnormal materials and N-triple materials), all other matters concerning the representation of general solutions are reduced to evaluation of algebraic expressions.

The Stroh formalism is often presented in terms of six-dimensional eigenmatrices and eigenvectors. In this form, the formalism does not suggest appropriate analytical expressions of the eigenvectors **a** and **b** (as functions of μ) which, in the case of a repeated eigenvalue, may produce the generalized eigenvectors according to the derivative rule. It also does not directly yield three sets of explicit expressions of the Barnett–Lothe tensors depending on the multiplicity of eigenvalues. The usual presentation of the subject is further complicated, not simplified, by using *normalized* eigenvectors. Analytical results and conclusions in this formalism have been obtained mostly for the non-degenerate case and occasionally for degenerate materials by modifying the proof. In the following, we obtain certain key expressions in the six-dimensional formalism through a unified derivation comprising all cases of material degeneracy.

For an eigenvalue μ and associated eigenvectors **a** and **b**, Eqs. (4) and (7b) yield

$$-(\mathbf{R}^{\mathrm{T}} + \mu\mathbf{T})\mathbf{a} = \mathbf{b}, \qquad (\mathbf{Q} + \mu\mathbf{R})\mathbf{a} = \mu\mathbf{b}. \tag{72a,b}$$

These equations may be rewritten in the form

$$\mathbf{N}\xi = \mu\xi, \qquad \mathbf{N}^{\mathrm{T}}\eta = \mu\eta. \tag{73a,b}$$

where

$$\mathbf{N} \equiv \begin{bmatrix} -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} & -\mathbf{T}^{-1} \\ \mathbf{Q} - \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} & -\mathbf{R}\mathbf{T}^{-1} \end{bmatrix}, \qquad \xi \equiv \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}, \qquad \eta \equiv \begin{Bmatrix} \mathbf{b} \\ \mathbf{a} \end{Bmatrix} \tag{74a,b,c}$$

Since **N** is a real matrix, if μ is an eigenvalue with the eigenvector ξ then $\bar{\mu}$ is also an eigenvalue with the associated eigenvector $\bar{\xi}$. **N** is called simple if it has three distinct complex conjugate pairs of eigenvalues. It is called non-semisimple if there is a repeated eigenvalue with only one independent eigenvector. Otherwise **N** is called semisimple. It is clear that the matrix **N** of a normal material is either simple or non-semisimple, and that of an abnormal material is semisimple. Unlike the preceding formalisms based on three-dimensional eigenrelations, the six-dimensional Stroh formalism does not imply straightforwardly the impossibility of three independent complex conjugate pairs of eigenvectors associated with a triple eigenvalue of **N** (i.e., the impossibility for **N** to be "extraordinary semisimple"), unless the material has a non-positive definite strain energy. A proof was given by Ting (1997). In the present analysis, the conclusion follows trivially because the eigenmatrix $\mathbf{M}(\mu)$ is two-dimensional. If μ is a double eigenvalue with only one independent pair of eigenvectors (N-double materials), then Eqs. (10), (29) and (31) yield

$$-\left(\mathbf{R}^{\mathrm{T}}+\mu\mathbf{T}\right)\mathbf{a}^{*}-\mathbf{T}\mathbf{a}=\mathbf{b}^{*},\qquad (\mathbf{Q}+\mu\mathbf{R})\mathbf{a}^{*}+\left\{\mathbf{R}+\mathbf{R}^{\mathrm{T}}+\mu\mathbf{T}\right)\mathbf{a}=\mu\mathbf{b}^{*}, \tag{75a,b}$$

which may be rewritten as

$$\mathbf{N}\begin{Bmatrix}\mathbf{a}^{*}\\ \mathbf{b}^{*}\end{Bmatrix}\equiv\mathbf{N}\xi^{*}=\mu\xi^{*}+\xi,\qquad \mathbf{N}^{\mathrm{T}}\begin{Bmatrix}\mathbf{b}^{*}\\ \mathbf{a}^{*}\end{Bmatrix}\equiv\mathbf{N}^{\mathrm{T}}\eta^{*}=\mu\eta^{*}+\eta. \tag{76a,b}$$

If μ is a triple eigenvalue with only one independent pair of eigenvectors (N-triple material), then Eqs. (10), (35), (38) and (75a) yield

$$-\left(\mathbf{R}^{\mathrm{T}}+\mu\mathbf{T}\right)\mathbf{a}^{**}-2\mathbf{T}\mathbf{a}^{*}=\mathbf{b}^{**},\qquad (\mathbf{Q}+\mu\mathbf{R})\mathbf{a}^{*}+2\mathbf{R}\mathbf{a}^{*}-2\mathbf{b}^{*}=\mu\mathbf{b}^{**}, \tag{77a,b}$$

or

$$\mathbf{N}\begin{Bmatrix}\mathbf{a}^{**}\\ \mathbf{b}^{**}\end{Bmatrix}\equiv\mathbf{N}\xi^{**}=\mu\xi^{**}+2\xi^{*},\qquad \mathbf{N}^{\mathrm{T}}\begin{Bmatrix}\mathbf{b}^{**}\\ \mathbf{a}^{**}\end{Bmatrix}\equiv\mathbf{N}^{\mathrm{T}}\eta^{**}=\mu\eta^{**}+2\eta^{*}. \tag{78a,b}$$

The matrix **N** does not suggest appropriate analytical expressions of **a** and **b** in terms of μ for the application of the derivative rule. Hence the generalized eigenvectors cannot be obtained by differentiation and must be determined by solving Eqs. (76a,b) and (78a,b). However, if **a** is chosen to be a column vector of $\hat{\Gamma}(\mu)$ (or of $\hat{\Gamma}'(\mu)$ if $\hat{\Gamma}(\mu)$ vanishes), as suggested in Section 3, then, with $\mathbf{b}=-(\mathbf{R}^{\mathrm{T}}+\mu T)\mathbf{a}$, the derivative rule applies and Eqs. (76a,b) and (78a,b) may be written as

$$\mathrm{d}/\mathrm{d}\mu\left[(\mathbf{N}-\mu\mathbf{I})\xi\right]=\mathbf{0},\qquad \mathrm{d}^2/\mathrm{d}\mu^2\left[(\mathbf{N}-\mu\mathbf{I})\xi\right]=\mathbf{0},$$

$$\mathrm{d}/\mathrm{d}\mu\left[\left(\mathbf{N}^{\mathrm{T}}-\mu\mathbf{I}\right)\eta\right]=\mathbf{0},\qquad \mathbf{d}^2/\mathbf{d}\mu^2\left[\left(\mathbf{N}^{\mathrm{T}}-\mu\mathbf{I}\right)\eta\right]=\mathbf{0}.$$

For a non-degenerate material, let **P** be the diagonal matrix whose elements are the eigenvalues μ_i (i = 1, 2, 3) with positive imaginary parts and let $\mathbf{A}=\{\mathbf{a}(\mu_1),\ \mathbf{a}(\mu_2),\ \mathbf{a}(\mu_3)\}$ and $\mathbf{B}=\{\mathbf{b}(\mu_1),\ \mathbf{b}(\mu_2),\ \mathbf{b}(\mu_3)\}$. For the degenerate and extra-degenerate case, **P** is taken to be, respectively,

$$\begin{bmatrix}\hat{\mu} & 0 & 0\\ 0 & \mu_0 & 1\\ 0 & 0 & \mu_0\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}\mu_0 & 1 & 0\\ 0 & \mu_0 & 2\\ 0 & 0 & \mu_0\end{bmatrix}$$

(where $\hat{\mu}=\mu_0$ for A-triple materials). For the degenerate case define $\mathbf{A}=\{\mathbf{a}(\hat{\mu}),\ \mathbf{a}(\mu_0),\ \mathbf{a}^{*}(\mu_0)\}$ and $\mathbf{B}=\{\mathbf{b}(\hat{\mu}),\ \mathbf{b}(\mu_0),\ \mathbf{b}^{*}(\mu_0)\}$ whereas for the extra-degenerate case $\mathbf{A}=\{\mathbf{a}(\mu_0),\ \mathbf{a}^{*}(\mu_0),\ \mathbf{a}^{**}(\mu_0)\}$ and $\mathbf{B}=\{\mathbf{b}(\mu_0),\ \mathbf{b}^{*}(\mu_0),\ \mathbf{b}^{**}(\mu_0)\}$. Eqs. (73a,b), (76a,b) and (78a,b) yield

$$\mathbf{N}\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}=\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}\mathbf{P},\qquad \mathbf{N}^{\mathrm{T}}\begin{bmatrix}\mathbf{B}\\ \mathbf{A}\end{bmatrix}=\begin{bmatrix}\mathbf{B}\\ \mathbf{A}\end{bmatrix}\mathbf{P}^{\mathrm{T}} \tag{79a,b}$$

Since **N** is a real matrix, one has

$$\mathbf{N}\begin{bmatrix}\mathbf{A} & \bar{\mathbf{A}}\\ \mathbf{B} & \bar{\mathbf{B}}\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \bar{\mathbf{A}}\\ \mathbf{B} & \bar{\mathbf{B}}\end{bmatrix}\begin{bmatrix}\mathbf{P} & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{P}}\end{bmatrix},\qquad \mathbf{N}^{\mathrm{T}}\begin{bmatrix}\mathbf{B} & \bar{\mathbf{B}}\\ \mathbf{A} & \bar{\mathbf{A}}\end{bmatrix}=\begin{bmatrix}\mathbf{B} & \bar{\mathbf{B}}\\ \mathbf{A} & \bar{\mathbf{A}}\end{bmatrix}\begin{bmatrix}\mathbf{P}^{\mathrm{T}} & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{P}}^{\mathrm{T}}\end{bmatrix} \tag{80a,b}$$

Eq. (22) and $\Omega = \mathbf{A}^{\mathrm{T}}\mathbf{B} + \mathbf{B}^{\mathrm{T}}\mathbf{A}$ yield

$$\begin{bmatrix}\mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}}\\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{A} & \bar{\mathbf{A}}\\ \mathbf{B} & \bar{\mathbf{B}}\end{bmatrix}=\begin{bmatrix}\Omega & \mathbf{0}\\ \mathbf{0} & \bar{\Omega}\end{bmatrix} \tag{81}$$

and the closure relations (see Section 3 of Part I)

$$\begin{bmatrix}\mathbf{A} & \bar{\mathbf{A}}\\ \mathbf{B} & \bar{\mathbf{B}}\end{bmatrix}\begin{bmatrix}\Omega^{-1} & \mathbf{0}\\ \mathbf{0} & \bar{\Omega}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}}\\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}}\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix} \tag{82}$$

The last equation and Eq. (80a) yield

$$\mathbf{N}=\begin{bmatrix}\mathbf{A} & \bar{\mathbf{A}}\\ \mathbf{B} & \bar{\mathbf{B}}\end{bmatrix}\begin{bmatrix}\mathbf{P}\Omega^{-1} & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{P}}\bar{\Omega}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}}\\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}}\end{bmatrix} \tag{83}$$

Using Eq. (49a), one obtains

$$\mathbf{N}^{*}\mathbf{N}=\mathbf{N}\mathbf{N}^{*}=\begin{bmatrix}\mathbf{A} & \bar{\mathbf{A}}\\ \mathbf{B} & \bar{\mathbf{B}}\end{bmatrix}\begin{bmatrix}-i\mathbf{P}\Omega^{-1} & \mathbf{0}\\ \mathbf{0} & i\bar{\mathbf{P}}\bar{\Omega}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}}\\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}}\end{bmatrix} \tag{84}$$

For the case of simple or semisimple $\mathbf{N}$, relations similar to Eqs. (49a), (73a,b), (76a,b), (80a,b), (83) and (84) have been given in terms of normalized eigenvectors (Ting, 1996a, 1996b), for which $\Omega = \mathbf{I}$. The present equations, in terms of unnormalized eigenvectors and generalized eigenvectors, are valid for all types of anisotropic materials.

9. Summary and conclusion

Formulation of the eigensolutions in terms of the elastic stiffness $[\beta]^{-1}$ yields the same classification of anistropic materials into five mutually exclusive types. The eigenvalues are the roots of the characteristic equation $\Delta(\mu) \equiv |\Gamma(\mu)| = 0$. For each eigenvalue μ, the **a**-vector is determined first by choosing a nontrivial column vector of the adjoint matrix $\hat{\Gamma}(\mu)$. For an abnormal material there is an eigenvalue with $\hat{\Gamma}(\mu) = \mathbf{0}$. Then $\hat{\Gamma}'(\mu)$ does not vanish and it contains one or two independent columns to be chosen as the **a**-vectors. The **b**-vector associated with an **a**-vector is given by Eq. (16). If $\Delta(\mu) \equiv 0$ has a repeated root and the number of independent eigenvectors is less than three, additional eigensolutions may be obtained in the form of Eqs. (27) and (36), where the generalized eigenvectors are obtained from the eigenvectors by the derivative rule.

The transformation rules of Eq. (7a,b) are valid for eigenvectors only and not for generalized eigenvectors. These transformation rules, along with the derivative rule, map the eigenspace of the **a**-vectors (and generalized **a**-vectors) in the Stroh formalism into the eigenspace of the **b**-vectors (and generalized **b**-vectors) in the Lekhnitskii formalism, and vice versa. This dual relation is based on Eq. (3) and its generalizations to the degenerate and extra-degenerate cases, Eqs. (28) and (37). The three equations immediately yield the eigenrelations of Eqs. (4), (29) and (35) in the Stroh formalism when one chooses to work with the stiffness matrix $[\beta]^{-1}$; they provide the eigenrelations of the Lekhnitskii formalism (i.e., Eq. (5) of this part and (4.5) and (5.5) of Part I) if one uses instead the reduced compliance matrix $[\beta]$. The latter set of eigenrelations show the latent structure of the Lekhnitskii formalism that was originally found and used in a singularity analysis of multi-material wedges (Yin, 1997). By using the two sets of eigenrelations, analytical expressions of all eigenvectors may be obtained in terms of the eigenvalues and the elements of $[\beta]^{-1}$ or of $[\beta]$. However, the expressions are much simpler in the Lekhnitskii formalism. A concise proof of the equivalence of the dual formalisms, including the equivalence of the characteristic equations $\delta(\mu) = 0$ and $\Delta(\mu) = 0$, is given in Section 7.The eigenvectors in each formalism satisfies the modified orthogonality and closure relations [Eqs. (81) and (82)]. These relations contain the matrix $\Omega \equiv \mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B}$, whose algebraic form depends on the five types of anisotropic materials. The Barnett–Lothe tensors may be obtained, in both Lekhnitskii and Stroh formalisms and for all types of materials according to the same Eq. (24), in terms of Ω^{-1} and the respective sets of **a**- and **b**-vectors (based, respectively, on $[\beta]$ and $[\beta]^{-1}$). We notice that, in the existing literature, normality and closure relations as well as the Barnett–Lothe tensors are expressed in terms of *normalized* eigenvectors for which Ω reduces to the identity tensor. Such relations are *not* valid in the degenerate and extra-degenerate cases. However, with the inclusion of the matrix Ω and its various expressions for different types of materials, a significant number of important relations in anisotropic elasticity assume universally valid forms independent of material degeneracy. These include the modified orthogonality and closure relations, Eqs. (24), (49a), (49b), (83) and (84). Some new identities relating the Barnett–Lothe tensors and the elasticity matrices are given as Eqs. (65)–(69).

Although the eigenrelations establish a true dualism between the two formalisms, an asymmetry is introduced in the dualism by the relation $b_1 = -\mu b_2$, which causes the **b**-vectors to be determined by a 2 $\times$ 2 eigenmatrix $\mathbf{M}(\mu)$. Thus the various expression in the Lekhnitskii formalism involve the 2 $\times$ 2 adjoint matrix $\mathbf{U}(\mu)$ and its derivatives, whereas those in the Stroh formalism involve the 3 $\times$ 3 matrix $\hat{\Gamma}(\mu)$ and its derivatives. Notice that each element of $\hat{\Gamma}(\mu)$ is the determinant of a 2 $\times$ 2 stiffness matrix. While the resulting expressions of the Barnett–Lothe tensors in the two formalisms have analogous matrix forms, the detailed expressions are significantly more complicated in the Stroh formalism. The same is true for the general solutions of the displacements and the stress, particularly in the degenerate and extra-degenerate cases, where the generalized eigensolutions may be obtained by differentiating analytical expressions of the eigensolutions. It appears that most of the significant theoretical results on general solutions of elastostatics previously obtained in the Stroh formalism (i.e., in terms of the elements of $[\beta]^{-1}$) can be obtained in simpler forms, and in a simpler way, by using the Lekhnitskii formalism.

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